Conservation Laws and Symmetries of Semilinear Radial Wave Equations

Stephen C. Anco

Department of Mathematics, Brock University, St. Catharines, ON, L2S 3A1, Canada

Nataliya M. Ivanova

Institute of Mathematics of NAS of Ukraine, 01601 Kyiv, Ukraine
and
Department of Mathematics, University of British Columbia, Vancouver, BC, V6T 1Z2, Canada

Abstract

Classifications of symmetries and conservation laws are presented for a variety of physically and analytically interesting wave equations with power nonlinearities in n spatial dimensions: a radial hyperbolic equation, a radial Schrödinger equation and its derivative variant, and two proposed radial generalizations of modified Korteweg–de Vries equations, as well as Hamiltonian variants. The mains results classify all admitted local point symmetries and all admitted local conserved densities depending on up to first order spatial derivatives, including any that exist only for special powers or dimensions. All such cases for which these wave equations admit, in particular, dilational energies or conformal energies and inversion symmetries are determined. In addition, potential systems arising from the classified conservation laws are used to determine nonlocal symmetries and nonlocal conserved quantities admitted by these equations. As illustrative applications, a discussion is given of energy norms, conserved H^s norms, critical powers for blow-up solutions, and one-dimensional optimal symmetry groups for invariant solutions.

Key words: semilinear wave equation, conservation laws, symmetries, invariant solutions, conserved energy, critical power, Hamiltonian, NLS equation, KdV equation
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Email addresses: sanco@brocku.ca (Stephen C. Anco), ivanova@imath.kiev.ua (Nataliya M. Ivanova).

1 Introduction

Over past few decades there has been a lot of work on global analysis of nonlinear wave equations in $n \geq 1$ spatial dimensions [24],

$$u_{tt} = \Delta u + f(x, u, \nabla u), \quad u(t,x) \in \mathbb{R}, \tag{WEa}$$

$$\mathrm{i}u_t = \Delta u + f(x, |u|, |\nabla u|)u, \quad u(t,x) \in \mathbb{C}, \tag{WEb}$$

with one main focus being the study of blow-up phenomenon for the case of power nonlinearities $f = \pm|u|^p$, $\pm|\nabla u|^p$. Some key tools with a number of uses in this study are conservation laws and symmetries.

Conservation laws such as energy provide basic conserved quantities used in obtaining estimates on $|u|$ or $|\nabla u|$ for smooth solutions, and also in defining suitable norms for weak solutions. Of considerable interest are extra conservation laws such as conformal energies that can appear for special powers p depending on the dimension n. Symmetries, in contrast, lead to exact group-invariant solutions and play a role in defining invariant Sobolev norms. Scaling symmetries are of special relevance, as the critical nonlinearity power for blow-up is typically singled out by scaling-invariance of a positive energy norm. Moreover, scaling transformation arguments give a means of relating the behavior of solutions in different regimes, for instance, solutions at short times with large initial data can be scaled to long times with small initial data when the nonlinearity power is subcritical.

In this paper we present classifications of conservation laws and symmetries admitted by a variety of physically and analytically interesting semilinear radial wave equations. The equations will be organized according to their variational structure as follows.

First, we consider the standard (hyperbolic) nonlinear wave equation

$$u_{tt} = \Delta u \pm u^p \tag{NLW}$$

as well as the nonlinear Schrödinger equation

$$\mathrm{i}u_t = \Delta u \pm |u|^p u \tag{NLS}$$

and its radial derivative variant

$$\mathrm{i}u_t = \Delta u \pm \mathrm{i}|u|^p\Big(u_r + \frac{m}{p+2}r^{-1}u\Big) \tag{dNLS}$$

where $\Delta = r^{-m}\partial_r r^m \partial_r = \partial_r^2 + mr^{-1}\partial_r$ is the radial Laplacian in $n = m+1 \geq 1$ (spatial) dimensions. These PDEs each arise as the stationary points $\delta\mathcal{L}/\delta u =$

0 of a Lagrangian functional $\mathcal{L} = \int_{-\infty}^{+\infty} \int_0^\infty L[u] r^m dr dt$, given by

$$L_{\rm NLW}[u] = \frac{1}{2}(-u_t^2 + u_r^2) \mp \frac{1}{p+1} u^{p+1},$$
$$L_{\rm NLS}[u] = {\rm i}\bar{u}u_t + |u_r|^2 \mp \frac{1}{p+2}|u|^{p+2},$$
$$L_{\rm dNLS}[u] = {\rm i}\bar{u}u_t + |u_r|^2 \mp \frac{\rm i}{p+2}|u|^p(\bar{u}u_r - u\bar{u}_r).$$

As is well known, the NLS equation also possesses a Hamiltonian formulation $u_t = {\rm i}\delta\mathcal{H}/\delta\bar{u}$ with $\mathcal{H} = \int_0^\infty H[u] r^m dr$ given by

$$H_{\rm NLS}[u] = -|u_r|^2 \pm \frac{2}{p+2}|u|^{p+2}$$

where multiplication by i defines a Hamiltonian operator with respect to the L^2 Hermitian inner product on the radial line $0 \leq r < \infty$. Next, while the derivative version (dNLS) of the NLS equation is not Hamiltonian, it does have a Hamiltonian variant

$$ {\rm i}u_t = \Delta u + \frac{m(m-2)}{4} r^{-2} u \pm {\rm i}\left((|u|^p u)_r + \frac{m}{2}|u|^p r^{-1} u \right) \tag{dNLS-H}$$

which reduces to the standard derivative NLS equation in the case $m = 0$. This radial generalization arises from the Hamiltonian formulation

$$u_t = r^{-m/2}\partial_r(r^{-m/2}\delta\mathcal{H}/\delta\bar{u}), \quad \mathcal{H} = \textstyle\int_0^\infty H[u] r^m dr \tag{1}$$

with

$$H_{\rm dNLS}[u] = \frac{\rm i}{2}(u\bar{u}_r - \bar{u}u_r) \pm \frac{2}{p+2}|u|^{p+2}.$$

Here $r^{-m/2}\partial_r r^{-m/2}$ is easily verified to be a Hamiltonian operator with respect to the radial L^2 Hermitian inner product. In particular, such an operator $\mathcal{D}$ with no dependence on u or $\bar{u}$ or their derivatives is *Hamiltonian* [20] iff it is skew-adjoint in this inner product so consequently the Poisson bracket associated to it by $\{\mathcal{P}, \mathcal{Q}\}_\mathcal{D} = \int_0^\infty (\delta\mathcal{P}/\delta\bar{u})\mathcal{D}(\delta\mathcal{Q}/\delta u) dr$ will be skew-Hermitian and obey the Jacobi identity for all real functionals $\mathcal{P} = \int_0^\infty P[u] r^m dr$, $\mathcal{Q} = \int_0^\infty Q[u] r^m dr$. Note that, as a consequence of the skew-adjoint property, the Hamiltonian $\mathcal{H} = \int_0^\infty H[u] r^m dr$ will formally be a conserved quantity, $\frac{d}{dt}\mathcal{H} = 0$ (to within boundary terms at spatial infinity), for all formal solutions u.

Last, based on the factorization of the Laplacian $\Delta = (\partial_r + m r^{-1})\partial_r$, we propose two radial generalizations of the modified Korteweg–de Vries equation,

$$u_t = (\Delta u \pm u^{p+1})_r \tag{mKdV-1}$$

and

$$u_t = (\Delta u \pm u^{p+1})_r + \frac{m}{r}(\Delta u \pm u^{p+1}) \tag{mKdV-2}$$

both of which have neither a Lagrangian nor Hamiltonian formulation except in the case $m = 0$. We also introduce a Hamiltonian variant given by

$$u_t = (\Delta u \pm u^{p+1})_r + \frac{m}{2r}(\Delta u \pm u^{p+1}) \tag{mKdV-H}$$

with

$$H_{\mathrm{mKdV}}[u] = -\frac{1}{2}u_r^2 \pm \frac{1}{p+2}u^{p+2}$$

using the previous Hamiltonian operator (1), specialized in the obvious way to real functions $u = \bar{u}$. These radial mKdV equations are examples of third order evolutionary wave equations

$$u_t = \Delta\nabla_x u + f(x, u, \nabla u, \Delta u), \quad u(t, x) \in \mathbb{R}, \tag{WEc}$$

with power nonlinearities $f = u^p \nabla_x u$, u^{p+1}, where $\nabla_x = |x|^{-1} x \cdot \nabla$ is the radial gradient. Well-posedness of such wave equations is an interesting problem which has received some recent attention.

We emphasize the wave equations (dNLS), (dNLS-H), (mKdV-1,2,H) for $m \neq 0$ are new, being radial generalizations of the familiar ($n = 1$ dimensional) derivative Schrödinger equation and modified Korteweg–de Vries equation.

To begin we recall the definitions of symmetries and conservation laws from an analytical perspective (see also [20,5]). A *point symmetry* of a radial wave equation (WE) is a group of transformations given by an infinitesimal generator

$$\delta t = \tau(t, r, u), \qquad \delta r = \xi(t, r, u), \qquad \delta u = \eta(t, r, u)$$

on the variables t, r, u in the real-valued case, or

$$\delta t = \tau(t, r, u, \bar{u}), \quad \delta r = \xi(t, r, u, \bar{u}), \quad \delta u = \eta(t, r, u, \bar{u}), \quad \delta\bar{u} = \bar{\eta}(t, r, u, \bar{u})$$

on the variables t, r, u and $\bar{u}$ in the complex-valued case, such that the equation (WE) is preserved. On solutions, such a transformation in both cases is infinitesimally equivalent to

$$\delta t = \delta r = 0, \qquad \delta u = \eta - \tau u_t - \xi u_r \tag{2}$$

called the *characteristic form* of the point symmetry. The expressions η, τ, ξ are determined by the Fréchet derivative of the wave equation (WE) applied to δu holding for all formal solutions u. (More precisely, one works in a jet space setting, using the coordinate space defined by t, r, u and all derivatives of u modulo the equation (WE) and its differential consequences. In jet space, a point symmetry is the prolongation of the operator $X = \tau\partial_t + \xi\partial_r + \eta\partial_u$. When u is complex-valued, jet space is enlarged in the obvious way by $\bar{u}$ and its derivatives.) The set of all infinitesimal point symmetries admitted by a given wave equation (WE) has the structure of a Lie algebra (for the operators

X under commutation). For a given point symmetry, invariant solutions are characterized by the form $\delta u = \eta(t,r,u) - \tau(t,r,u)u_t - \xi(t,r,u)u_r = 0$ when $u(t,r) \in \mathbb{R}$, or $\delta u = \eta(t,r,u,\bar{u}) - \tau(t,r,u,\bar{u})u_t - \xi(t,r,u,\bar{u})u_r = 0$ together with its complex conjugate when $u(t,r) \in \mathbb{C}$.

A *conservation law* of a radial wave equation (WE) is given by a space-time divergence $D_t(r^m\Psi^t) + D_r(r^m\Psi^r)$ that is equal to a linear combination of the equation and its differential consequences, so that

$$D_t\Psi^t + D_r\Psi^r + mr^{-1}\Psi^r = 0 \tag{3}$$

holds for all formal solutions u. The radial integral of the conserved density Ψ^t formally satisfies

$$\frac{d}{dt}\int_0^\infty \Psi^t r^m dr = -r^m\Psi^r|_0^\infty = 0$$

which vanishes when the flux Ψ^r at spatial infinity is zero or decays faster than r^{-m}. Hence $C = \int_0^\infty \Psi^t r^m dr$ formally yields a conserved quantity for the equation (WE). Conversely, any such conserved quantity arises from a conservation law (3). Two conservation laws are equivalent if their conserved densities Ψ^t differ by a radial divergence $r^{-m}D_r(r^m\Theta)$ on all formal solutions u, giving the same conserved quantity C up to boundary terms. The set of all conservation laws (up to equivalence) admitted by a given wave equation (WE) forms a vector space, on which there is a natural action [9] by the Lie group of all admitted point symmetries of the equation (WE).

Each conservation law (3) of a radial wave equation (WE) has an equivalent *characteristic form* where $D_t(r^m\Psi^t) + D_r(r^m\Psi^r)$ is just proportional to the equation (WE) multiplied by an expression Q that depends on the jet variables (up to some finite differential order). Such expressions Q for which the product of equation (WE) and Q yields a total space-time divergence (and hence a conservation law on solutions of the equation (WE)) are called *multipliers*. There is a specific relation between multipliers Q and conserved densities Ψ^t: in the case of a hyperbolic wave equation (WEa), conserved densities $\Psi^t(t,r,u,u_r,u_t)$ modulo radial divergences correspond to multipliers

$$Q(t,r,u,u_r,u_t) = \delta(\Psi^t r^m)/\delta u_t, \tag{4}$$

while in the case of an evolutionary wave equation (WEb) and (WEc), conserved densities $\Psi^t(t,r,u,u_r)$ or $\Psi^t(t,r,u,\bar{u},u_r,\bar{u}_r)$ modulo radial divergences correspond to multipliers

$$Q(t,r,u,u_r,u_{rr}) = \delta(\Psi^t r^m)/\delta u, \tag{5}$$

$$Q(t,r,u,\bar{u},u_r,\bar{u}_r,u_{rr},\bar{u}_{rr}) = -\mathrm{i}\delta(\Psi^t r^m)/\delta u, \tag{6}$$

respectively. In all cases the multiplier Q is determined by [2,3] the adjoint of the Fréchet derivative of equation (WE) applied to Q, augmented by additional equations formed from the Fréchet derivative of Q itself, holding for all

formal solutions u. (More precisely, one works in the same jet space as for the computation of symmetries.) Thus the determination of conservation laws via multipliers is a kind of adjoint problem [1] of the determination of symmetries.

If a wave equation (WE) possesses a Lagrangian formulation, then its multipliers Q define *variational* symmetries $\delta t = \delta r = 0$, $\delta u = Q$ in the real-valued case or $\delta t = \delta r = 0$, $\delta u = \bar{Q}$ in the complex-valued case, such that the determining equations on Q reduce to conditions equivalent to those given by Noether's theorem [20,2,3] for the Lagrangian to be formally invariant (to within boundary terms at spatial infinity). Variational symmetries corresponding to multipliers (4), (5), (6) for conserved densities containing derivatives of u or $\bar{u}$ will be a point symmetry $\delta u = Q$ in the real-valued case only if the derivatives of u in Q at most are first order and appear linearly, or in the complex-valued case $\delta u = \bar{Q}$ will be a point symmetry only if the same is true for derivatives of $\bar{u}$ in Q and there are no derivatives of u in Q.

For each of the wave equations (NLW), (NLS), (dNLS) and (dNLS-H), (mKdV-1,2,H), we now classify all admitted conserved quantities containing up to first order spatial derivatives, along with all admitted point symmetries. In these classifications, the nonlinearity power will be restricted to $p \neq 0, 1$ for (NLW), and $p \neq 0$ for (NLS), (dNLS), (dNLS-H), (mKdV-1,2,H), so that all cases of linear wave equations are excluded. No restrictions will be placed on m (even allowing non-integer values). All computations have been carried out using the computer algebra programs LiePDE and ConLaw [25,26] utilizing an enhanced version of the program Crack [27] for solving overdetermined systems of equations containing parameters.

Throughout we use the notation $F(u,q) = \begin{cases} \frac{1}{q}u^q, & q\neq 0 \\ \ln u, & q=0 \end{cases}$. Also, a "±" sign will refer to the sign of the nonlinear term in the wave equations, usually called the *focusing/defocusing* cases, respectively.

2 Point symmetries

All of the wave equations (NLW), (NLS), (dNLS) and (dNLS-H), (mKdV-1,2,H) obviously admit time translations $\tau = 1$, $\xi = \eta = 0$, and, for $m = 0$, space translations $\xi = 1$, $\tau = \eta = 0$. In addition the Schrödinger equations (NLS), (dNLS), (dNLS-H) admit phase rotations $\tau = \xi = 0$, $\eta = \mathrm{i}u$.

The following tables list, firstly, the scaling symmetries admitted by these equations, and secondly, any extra admitted point symmetries. (Note these classifications exclude all linear cases i.e. $p = 0$; plus $p = 1$ for (NLW).)

Table 1. Scaling symmetries

τ	ξ	η	Equation
t	r	$-\dfrac{2}{p-1}u$	(NLW)
$2t$	r	$-\dfrac{2}{p}u$	(NLS)
$2t$	r	$-\dfrac{1}{p}u$	(dNLS), (dNLS-H)
$3t$	r	$-\dfrac{2}{p}u$	(mKdV-1,2,H)

Table 2. Extra point symmetries for (NLW)

τ	ξ	η	Remarks
r	t	0	Lorentz boost, $m = 0$
$t^2 + r^2$	$2tr$	$-\dfrac{4}{p-1}tu$	inversion, $m = \dfrac{4}{p-1}$

Table 3. Extra point symmetries for (NLS)

τ	ξ	η	Remarks
0	$2t$	$\mathrm{i}ru$	Galilean boost, $m = 0$
t^2	tr	$\left(\dfrac{\mathrm{i}}{4}r^2 - \dfrac{2}{p}t\right)u$	inversion, $m = -1 + \dfrac{4}{p}$

The inversion and boost symmetries of equations (NLW) and (NLS) are well known [24], so our classification in tables 2 and 3 mainly provides a completeness result that these two equations do not possess any additional symmetries for special nonlinearity powers.

Surprisingly, no extra symmetries are found to be admitted by equations (dNLS) and (dNLS-H).

Table 4. Extra point symmetries for (mKdV-1)

τ	ξ	η	Remarks
0	$2t$	∓ 1	Galilean boost, $m = 0$, $p = 1$

Table 5. Extra point symmetries for (mKdV-2)

τ	ξ	η	Remarks
0	$2t$	∓ 1	Galilean boost, $m = 0$, $p = 1$
t^2	$\dfrac{2}{3}tr$	$-\dfrac{1}{3}(4tu \pm r)$	inversion, $m = 1$, $p = 1$

Table 6. Extra point symmetries for (mKdV-H)

τ	ξ	η	Remarks
0	$2t$	∓ 1	Galilean boost, $m = 0$, $p = 1$
t^2	$\dfrac{2}{3}tr$	$-\dfrac{1}{3}(4tu \pm r)$	inversion, $m = 2$, $p = 1$

Like the (NLS) equation, the mKdV equations (mKdV-2,H) possess boosts and inversions as extra symmetries for special nonlinearity powers, while no extra symmetries are found to be admitted by equation (mKdV-1). Our results in tables 4, 5, 6 for these radial mKdV equations with $m > 0$ are new.

Note the powers for which the inversions exist are called a *conformal power*.

Table 7. Conformal powers

p	Equation
$1 + \dfrac{4}{m}$	(NLW), $m \neq 0$
$\dfrac{4}{m+1}$	(NLS), $m \neq -1$
1	(mKdV-2), $m = 1$; (mKdV-H), $m = 2$

As a summary of the main results, we list the point symmetry algebras found for $m > 0$, with their generators denoted by X_{trans}, X_{scal}, X_{inver}, X_{phase}. These algebras fall into four classes (with a suitable normalization of X_{scal} in each case): For equation (mKdV-1) as well as the non-conformal case of equations (NLW) and (mKdV-2,H), the admitted algebra is generated by the time-translation and scaling symmetries

$$[X_{\text{trans}}, X_{\text{scal}}] = X_{\text{trans}}.$$

In the conformal case of equations (NLW) and (mKdV-2,H) this algebra is enlarged by an inversion symmetry

$$[X_{\text{trans}}, X_{\text{inver}}] = 2X_{\text{scal}}, \qquad [X_{\text{scal}}, X_{\text{inver}}] = X_{\text{inver}}.$$

For the conformal (NLS) equation, its admitted algebra is a central extension of the algebra for the conformal (NLW) equation, as generated by the phase rotation symmetry

$$[X_{\text{phase}}, X_{\text{trans}}] = [X_{\text{phase}}, X_{\text{scal}}] = [X_{\text{phase}}, X_{\text{inver}}] = 0,$$

while the admitted algebra for the non-conformal (NLS) equation as well as for the (dNLS) and (dNLS-H) equations is a similar central extension of the algebra for the non-conformal (NLW) equation.

Table 8. Point symmetry groups

Generators X	Group $\mathcal{G}$	Equation
X_{trans}, X_{scal}	$U(1) \rtimes U(1)$ solvable	(mKdV-1), non-conformal (NLW), non-conformal (mKdV-2,H)
X_{trans}, X_{scal}, X_{inver}	$SL(2,\mathbb{R})$ semisimple	conformal (NLW), conformal (mKdV-2,H)
X_{trans}, X_{scal}, X_{phase}	$(U(1) \rtimes U(1)) \times U(1)$ solvable	non-conformal (NLS), (dNLS), (dNLS-H)
X_{trans}, X_{scal}, X_{inver}, X_{phase}	$SL(2,\mathbb{R}) \times U(1)$ centrally extended semisimple	conformal (NLS)

3 Optimal symmetry groups for invariant solutions

All one-dimensional point symmetry subgroups $\mathcal{G}_{(1)}$ determine corresponding group-invariant solutions $u(t,r)$ when the above wave equations are augmented by the invariant surface condition $Xu = 0$ where X is the symmetry generator of $\mathcal{G}_{(1)}$ in characteristic form. In particular, the invariant surface condition reduces a wave equation (WE) to an ODE [20,5]. Any two conjugate subgroups will give rise to reduced ODEs that are related by a conjugacy transformation in the full point symmetry group $\mathcal{G}$ acting on the invariant solutions $u(t,r)$ determined by each subgroup. Hence, up to the action of $\mathcal{G}$, all invariant solutions for a given wave equation can be obtained by selecting a one-dimensional subgroup in each conjugacy class of all admitted one-dimensional point symmetry subgroups $\mathcal{G}_{(1)}$. Such a selection is called an *optimal set of subgroups* [21].

Optimal subalgebras have been classified in the work presented in [22] for all three- and four-dimensional Lie algebras. Applying this classification to the point symmetry algebras of the wave equations under consideration, we have the following list of optimal one-dimensional subalgebras for finding all invariant solutions of equations (NLW), (NLS), (dNLS) and (dNLS-H), (mKdV-1,2,H) for $m > 0$:

Table 9. Optimal one-dimensional subalgebras
(a, b, c are arbitrary constants)

Generators	Equation
X_{trans}, X_{scal}	(mKdV-1), non-conformal (NLW), non-conformal (mKdV-2,H)
X_{trans}, X_{scal}, $X_{\text{inver}} + X_{\text{trans}}$	conformal (NLW), conformal (mKdV-2,H)
X_{phase}, $X_{\text{scal}} + aX_{\text{phase}}$, $X_{\text{trans}} + bX_{\text{scal}} + cX_{\text{phase}}$	non-conformal (NLS), (dNLS), (dNLS-H)
X_{phase}, X_{trans}, $X_{\text{trans}} \pm X_{\text{scal}}$, $X_{\text{scal}} + aX_{\text{phase}}$, $X_{\text{trans}} + X_{\text{inver}} + bX_{\text{phase}}$	conformal (NLS)

Invariant solutions for (NLW), (NLS), (dNLS) with $m \geq 0$ (i.e. in all dimensions $n \geq 1$) have been derived in [4,23,16,11,12,10,17]. We will present invariant solutions of (dNLS-H), (mKdV-1,2,H) for $m > 0$ (i.e. in dimensions $n > 1$) elsewhere. (The $m = 0$ case of (mKdV-1,2,H) is treated in [20,18,14,23]).

4 Local conservation laws

As the wave equations (NLW), (NLS) and (dNLS) each have a Lagrangian formulation, all their admitted variational point symmetries yield corresponding conserved quantities which are well known [24] for (NLW) and (NLS). In particular, time translation symmetry yields energy, space translation and boost symmetries yield momenta, and inversion symmetry yields conformal energy, while phase rotation symmetry yields charge. In addition, there is a special nonlinearity power for which the scaling symmetry becomes variational and yields a dilational energy. Our classification in the following tables provides a completeness result that no additional conserved quantities up to first order are admitted by these equations (NLW), (NLS) and (dNLS) for special nonlinearity powers (excluding all linear cases i.e. $p = 0$; plus $p = 1$ for (NLW)).

The remaining wave equations (dNLS-H) and (mKdV-1,2,H) do not have a Lagrangian formulation, so consequently their admitted conservation laws come from multipliers given by adjoint-symmetries [3] rather than symmetries. For the Hamiltonian equations (dNLS-H) and (mKdV-H), note the Hamiltonian itself provides one conservation law. Our results in the following tables for all these radial equations with $m > 0$ are new.

Table 10. Local conservation laws for (NLW)

Ψ^t	Ψ^r	Remarks
$\frac{1}{2}(u_t^2 + u_r^2) \mp F(u, p+1)$	$-u_t u_r$	energy
$u_t u_r$	$-\frac{1}{2}(u_t^2 + u_r^2) \mp F(u, p+1)$	momentum, $m = 0$
$\frac{1}{2}r(u_t^2 + u_r^2) + t u_t u_r \mp rF(u, p+1)$	$-\frac{1}{2}t(u_t^2 + u_r^2) - r u_t u_r$ $\mp tF(u, p+1)$	boost momentum, $m = 0$
$\frac{1}{2}t(u_t^2 + u_r^2) + r u_t u_r$ $+\frac{2}{p-1}u u_t \mp tF(u, p+1)$	$-\frac{1}{2}r(u_t^2 + u_r^2) - t u_t u_r$ $-\frac{2}{p-1}u u_r \mp rF(u, p+1)$	dilational energy, $m = \frac{4}{p-1}$
$\frac{1}{2}(t^2 + r^2)(u_t^2 + u_r^2) + 2tr u_t u_r - \frac{2}{p-1}u^2$ $+\frac{4}{p-1}tu u_t \mp \left(t^2 + r^2\right) F(u, p+1)$	$-tr(u_t^2 + u_r^2) - (t^2 + r^2)u_t u_r$ $-\frac{4}{p-1}tu u_r \mp 2trF(u, p+1)$	conformal energy, $m = \frac{4}{p-1}$

Table 11. Local conservation laws for (NLS)

Ψ^t	Ψ^r	Remarks
$\lvert u\rvert^2$	$\mathrm{i}(u_r\bar{u} - u\bar{u}_r)$	charge
$\frac{1}{2}\lvert u_r\rvert^2 \mp F(\lvert u\rvert, p+2)$	$-\frac{1}{2}(u_r\bar{u}_t - u_t\bar{u}_r)$	energy
$\mathrm{i}t(u_r\bar{u} - u\bar{u}_r) - r\lvert u\rvert^2$	$\mathrm{i}t(u\bar{u}_t - u_t\bar{u}) + \mathrm{i}r(u\bar{u}_r - u_r\bar{u})$ $+2t\lvert u_r\rvert^2 \pm 4tF(\lvert u\rvert, p+2)$	boost momentum, $m=0$
$\frac{\mathrm{i}}{2}(u_r\bar{u} - u\bar{u}_r)$	$\frac{\mathrm{i}}{2}(u\bar{u}_t - u_t\bar{u})$ $+\lvert u_r\rvert^2 \pm 2F(\lvert u\rvert, p+2)$	momentum, $m=0$
$\frac{1}{2}t\lvert u_r\rvert^2 + \frac{\mathrm{i}}{8}r(u_r\bar{u} - u\bar{u}_r)$ $\mp tF(\lvert u\rvert, p+2)$	$-\frac{1}{2}t(u_t\bar{u}_r + u_r\bar{u}_t) - \frac{1}{4}r\lvert u_r\rvert^2$ $+\frac{\mathrm{i}}{8}r(u_t\bar{u} - u\bar{u}_t) - \frac{1}{2p}(u\bar{u}_r + u_r\bar{u})$ $\mp\frac{1}{2}rF(\lvert u\rvert, p+2)$	dilational energy, $m = \frac{4}{p} - 1$
$\frac{1}{2}t^2\lvert u_r\rvert^2 - \frac{1}{8}r^2\lvert u\rvert^2$ $+\frac{\mathrm{i}}{4}tr(u_r\bar{u} - u\bar{u}_r)$ $\mp t^2F(\lvert u\rvert, p+2)$	$-\frac{1}{2}t^2(u_t\bar{u}_r + u_r\bar{u}_t) - \frac{1}{2}t\lvert u_r\rvert^2$ $+\frac{\mathrm{i}}{8}r^2(u_r\bar{u} - u\bar{u}_r) + \frac{\mathrm{i}}{4}tr(u_t\bar{u} - u\bar{u}_t)$ $-\frac{1}{2p}t(u_r\bar{u} + u\bar{u}_r) \mp trF(\lvert u\rvert, p+2)$	conformal energy, $m = \frac{4}{p} - 1$

Table 12. Local conservation laws for (dNLS)

Ψ^t	Ψ^r	Remarks
$\lvert u\rvert^2$	$\mathrm{i}u_r\bar{u} - \mathrm{i}u\bar{u}_r \mp 2F(\lvert u\rvert, p+2)$	charge
$\lvert u_r\rvert^2 \mp \mathrm{i}(u_r\bar{u} - u\bar{u}_r)\lvert u\rvert^{-2}F(\lvert u\rvert, p+2)$	$-(u_r\bar{u}_t + u_t\bar{u}_r)$ $\mp\mathrm{i}(u_t\bar{u} - u\bar{u}_t)\lvert u\rvert^{-2}F(\lvert u\rvert, p+2)$	energy
$\frac{\mathrm{i}}{2}(u_r\bar{u} - u\bar{u}_r)$	$\frac{\mathrm{i}}{2}(u\bar{u}_t - u_t\bar{u}) + \lvert u_r\rvert^2$	$m=0$
$\frac{\mathrm{i}}{2}(u_r\bar{u} - u\bar{u}_r) + 2t\lvert u_r\rvert^2$ $\mp 2\mathrm{i}t(u_r\bar{u} - u\bar{u}_r)\lvert u\rvert^{-2}F(\lvert u\rvert, p+2)$	$\frac{\mathrm{i}}{2}(u_t\bar{u} - u\bar{u}_t) - 2t(u_r\bar{u}_t + u_t\bar{u}_r)$ $\pm 2\mathrm{i}t(u_t\bar{u} - u\bar{u}_t)\lvert u\rvert^{-2}F(\lvert u\rvert, p+2)$ $-\frac{1}{p}(u_r\bar{u} + u\bar{u}_r) - r\lvert u_r\rvert^2$	dilational energy, $m = \frac{2}{p} - 1$

Table 13. Local conservation laws for (dNLS-H)

Ψ^t	Ψ^r	Remarks
$r^{-m/2}(u + \bar{u})$	$r^{-m/2}(\mathrm{i}(\bar{u}_r - u_r) \mp \lvert u\rvert^p(u + \bar{u}) + \frac{m\mathrm{i}}{2}r^{-1}(\bar{u} - u))$	
$r^{-m/2}\mathrm{i}(\bar{u} - u)$	$r^{-m/2}(u_r + \bar{u}_r \pm \mathrm{i}\lvert u\rvert^p(u - \bar{u}) + \frac{m}{2}r^{-1}(u + \bar{u}))$	
$\frac{\mathrm{i}}{2}(u\bar{u}_r - u_r\bar{u})$ $\pm 2F(\lvert u\rvert, p+2)$	$\frac{\mathrm{i}}{2}(u_t\bar{u} - u\bar{u}_t) - \lvert u_r\rvert^2 - \frac{m^2}{4}r^{-2}\lvert u\rvert^2$ $-\lvert u\rvert^{p+2} \mp \mathrm{i}\lvert u\rvert^p(u\bar{u}_r - u_r\bar{u})$	Hamiltonian
$\lvert u\rvert^2$	$\mathrm{i}(u\bar{u}_r - u_r\bar{u}) \mp 2(p+1)F(\lvert u\rvert, p+2)$	charge, $m=0$
$\frac{\mathrm{i}}{2}t(u\bar{u}_r - u_r\bar{u})$ $\pm 2tF(\lvert u\rvert, p+2) + \frac{r}{2}\lvert u\rvert^2$	$\frac{\mathrm{i}}{2}t(u_t\bar{u} - u\bar{u}_t) - t(\lvert u_r\rvert^2 + \frac{m^2}{4}r^{-2}\lvert u\rvert^2)$ $-t(\lvert u\rvert^{p+2} \pm \mathrm{i}\lvert u\rvert^p(u\bar{u}_r - u_r\bar{u}))$ $+\frac{\mathrm{i}}{2}r(u_r\bar{u} - u\bar{u}_r) \mp (p+1)rF(\lvert u\rvert, p+2)$	$m = \frac{2}{p} - 2$

Table 14. Local conservation laws for (mKdV-H), $m \neq 0$

Ψ^t	Ψ^r	Remarks
$r^{-m/2}u$	$-r^{-m/2}\left(u_{rr} + mr^{-1}u_r \pm u^{p+1}\right)$	mass
$\frac{1}{2}u_r^2 \mp F(u,p+2)$	$-u_t u_r + \frac{1}{2}u_{rr}^2 + mr^{-1}u_r u_{rr}$ $\pm u^{p+1}u_{rr} + \frac{m^2}{2}r^{-2}u_r^2 \pm mr^{-1}u^{p+1}u_r + \frac{1}{2}u^{2p+2}$	Hamiltonian
$\frac{3}{2}tu_r^2 - \frac{1}{2}ru^2 \mp 3F(u,p+2)$	$-3tu_t u_r + \frac{3}{2}tu_{rr}^2 + \frac{3}{2}tu^{2p+2} \pm 3tu^{p+1}u_{rr} + ruu_{rr}$ $-6\frac{p-2}{p}tr^{-1}u_r u_{rr} + 6\frac{(p-2)^2}{p^2}tr^{-2}u_r^2 - \frac{1}{2}ru_r^2$ $-2\frac{p+1}{p}uu_r \pm (p+1)rF(u,p+2)$	$m = \frac{4}{p} - 2$
$\frac{9}{2}t^2u_r^2 \mp 3t^2u^3 - 3tru^2 \mp$ $\mp 6tr^{-1}u \mp r^2 u$	$-9t^2r^{-1}u_r u_t + \frac{9}{2}t^2r^{-1}u_{rr}^2 + 18t^2r^{-2}u_r u_{rr} \pm 3r^{-1}u$ $\pm 9t^2r^{-1}u^2u_{rr} + 6tuu_{rr} \pm 6tr^{-2}u_{rr} + 6tr^{-2}u^2$ $+18t^2r^{-3}u_r^2 - 3tu_r^2 \pm 18t^2r^{-2}u^2u_r + r^2u^2$ $\pm 12tr^{-2}u_r \mp u_r + \frac{9}{2}t^2r^{-1}u^4 \pm 4tu^3 \pm ru_{rr}$	$m = 2$, $p = 1$

Table 15. Local conservation laws for (mKdV-1), $m \neq 0$

Ψ^t	Ψ^r	Remarks
$r^{-m}u$	$-r^{-m}(u_{rr} + mr^{-1}u_r \pm u^{p+1})$	mass

Table 16. Local conservation laws for (mKdV-2), $m \neq 0$

Ψ^t	Ψ^r	Remarks
$r^{-m}u$	$-r^{-m}(u_{rr} + mr^{-1}u_r \pm u^{p+1})$	mass
$\frac{1}{2}r^{1/2}u^2$	$-r^{1/2}uu_{rr} + \frac{1}{2}r^{1/2}u_r^2 - r^{-1/2}uu_r \mp \frac{3}{4}r^{1/2}u^4 \pm \frac{1}{8}r^{-3/2}u^2$	dilational momentum, $m = 3/2$, $p = 2$
$\frac{1}{2}ru^2$	$-ruu_{rr} + \frac{1}{2}ru_r^2 - 2uu_r \mp \frac{3}{4}ru^4$	dilational momentum, $m = 3$, $p = 2$

For completeness we mention that in the $m = 0$ case the well known mKdV conservation laws are listed in [19,18,20,2].

5 Norms and critical powers

The wave equations (NLW), (NLS), (dNLS) and (dNLS-H), (mKdV-H) possess both a scaling symmetry (which is uniform in m) and a conserved energy or Hamiltonian, so they each have an associated critical power with respect to the energy norm, $E[u] = \int_0^\infty e[u]r^m dr$, as shown in the following table.

Table 17. Energy norms

$E[u]$	critical power p	Remarks
$\int_0^\infty \Big(\frac{1}{2}(\lvert u_r\rvert^2 + \lvert u_t\rvert^2) \mp F(u, p+1)\Big) r^m dr$	$1 + \frac{4}{m-1}$	(NLW), $m \neq 1$
$\int_0^\infty \Big(\frac{1}{2}\lvert u_r\rvert^2 \mp F(\lvert u\rvert, p+1)\Big) r^m dr$	$\frac{4}{m-1}$	(NLS), $m \neq 1$
$\int_0^\infty \Big(\frac{1}{2}\lvert u_r\rvert^2 \mp \mathrm{i}(u\bar{u}_r - \bar{u}u_r)\lvert u\rvert^{-2}F(\lvert u\rvert, p+2)\Big) r^m dr$	$\frac{2}{m-1}$	(dNLS), $m \neq 1$
$\int_0^\infty \Big(\frac{\mathrm{i}}{2}(u\bar{u}_r - \bar{u}u_r) \pm 2F(\lvert u\rvert, p+2)\Big) r^m dr$	$\frac{2}{m}$	(dNLS-H), $m \neq 0$
$\int_0^\infty \Big(\frac{1}{2}\lvert u_r\rvert^2 \pm F(u, p+2)\Big) r^m dr$	$\frac{4}{m-1}$	(mKdV-H), $m \neq 1$

All these wave equations also possess dilational energies or dilational Hamiltonians for special nonlinearity powers p depending on m, in addition to the well known conformal energies admitted for the (NLW) and (NLS) equations in the case of conformal powers p (cf. table 7). Interestingly, the Hamiltonian mKdV equation (mKdV-H) admits a conformal energy in this case too.

Table 18. Dilational energies and dilation powers

Dilational energy	Dilation power p	Remarks
$\int_0^\infty \Big(te[u] + \big(ru_r + \frac{m}{2}u\big)u_t\Big) r^m dr$	$1 + \frac{4}{m}$	(NLW), $m \neq 0$
$\int_0^\infty \Big(te[u] + \frac{\mathrm{i}}{4}r(\bar{u}u_r - \bar{u}_r u)\Big) r^m dr$	$\frac{4}{m+1}$	(NLS), $m \neq -1$
$\int_0^\infty \Big(te[u] + \frac{\mathrm{i}}{4}r(\bar{u}u_r - \bar{u}_r u)\Big) r^m dr$	$\frac{2}{m+1}$	(dNLS), $m \neq -1$
$\int_0^\infty \Big(te[u] + \frac{1}{4}r\lvert u\rvert^2\Big) r^m dr$	$\frac{2}{m+2}$	(dNLS-H), $m \neq -2$
$\int_0^\infty \Big(te[u] - \frac{1}{6}ru^2\Big) r^m dr$	$\frac{4}{m+2}$	(mKdV-H), $m \neq -2$

Table 19. Conformal energies and conformal powers

Conformal energy	Conformal power p	Remarks
$\int_0^\infty \Big((t^2 + r^2)e[u] + 2t\big(ru_r + \frac{m}{2}u\big)u_t\Big) r^m dr$	$1 + \frac{4}{m}$	(NLW), $m \neq 0$
$\int_0^\infty \Big(t^2 e[u] + \frac{\mathrm{i}}{2}tr(\bar{u}u_r - \bar{u}_r u) - \frac{1}{4}r^2\lvert u\rvert^2\Big) r^m dr$	$\frac{4}{m+1}$	(NLS), $m \neq -1$
$\int_0^\infty \Big(t^2 e[u] - \frac{1}{3}\big(tru^2 + \frac{1}{2}tr^{-1}u + \frac{1}{3}r^2 u\big)\Big) r^2 dr$	1	(mKdV-H), $m = 2$

An interesting pattern in tables 18 and 19 is that when m is expressed in terms of p then the difference $m_{\text{crit.}} - m_{\text{dil.}}$ (for any fixed p) is equal to $\mathrm{ord}(\partial_r) - \mathrm{ord}(\partial_t) + 1 > 0$ where "ord" refers to the highest order of a specified derivative appearing in the wave equation. Accordingly, the dilation and conformal powers are subcritical in all cases.

Two other norms of analytical interest are the radial L^2 norm and the radial H^s norm given by $||u||_{L^2} = (\int_0^\infty |u|^2 r^m dr)^{1/2}$ and $||u||_{H^s} = (\int_0^\infty |\partial_r^s u|^2 r^m dr)^{1/2}$ for any positive integer s. The latter norm has a natural extension to all $s \geq 0$ defined in terms of the Fourier transform $\hat{u} = \int_{\mathbb{R}^{m+1}} u(t, |x|) \exp(-k \cdot x) d^{m+1}x$ such that u is in H^s iff $(1 + |k|^s)\hat{u}$ is in $L^2(\mathbb{R}^{m+1})$. The following tables list

the critical powers p for which these norms are scaling-invariant. Note in the case of the Schrödinger equations (NLS), (dNLS) and (dNLS-H), the L^2 norm coincides with the conserved charge.

Table 20. L^2 critical powers

critical power p	Remarks
$1+\frac{4}{m+1}$	(NLW)
$\frac{4}{m+1}$	(NLS), (mKdV-1,2,H)
$\frac{2}{m+1}$	(dNLS), (dNLS-H)

Table 21. H^s critical powers

critical power p	critical s	Remarks
$1+\frac{4}{m+1-2s}$	$\frac{m+1}{2}-\frac{2}{p-1}$	(NLW)
$\frac{4}{m+1-2s}$	$\frac{m+1}{2}-\frac{2}{p}$	(NLS), (mKdV-1,2,H)
$\frac{2}{m+1-2s}$	$\frac{m+1}{2}-\frac{1}{p}$	(dNLS), (dNLS-H)

6 Concluding remarks

The utility of symmetries and conservation laws can be extended by means of potential systems [8,6,7]. A *potential system* for a radial wave equation (WE) arises from any conservation law such that vanishing set of its multiplier, $Q = 0$, is contained in the set of all formal solutions u of the given equation. Potentiating such a conservation law yields the system

$$v_t = r^m \Psi^r, \qquad v_r = -r^m \Psi^t$$

whose solutions v up to shifts ($v \to v+c$ for an arbitrary constant c) are in one-to-one correspondence with the set of solutions u. For a given potential system, any admitted symmetry or conservation law that has an essential dependence on the potential v represents a *nonlocal symmetry* or *nonlocal conservation law*, respectively, of the wave equation (WE). Of course, the superposition of a local symmetry or a local conservation law with a nonlocal one yields further nonlocal ones, and so for the purpose of classifications we will mod out the admitted sets of local symmetries and conservation laws.

All potential systems arising from the conservation laws for the wave equations in tables 10 to 16 are given by potentiating: the mKdV equations themselves (mKdV-1,2,H) and the Hamiltonian variant of the derivative Schrödinger equation itself (dNLS-H); the charge conservation law for the Schrödinger equations (NLS), (dNLS), and (dNLS-H) in the case $m = 0$; and the dila-

tional momentum conservation laws for the mKdV equation (mKdV-2) in the cases $m = 3, \frac{3}{2}$. We find that only the first of these potential systems — potentiation of the (mKdV-1) equation itself (including the case $m = 0$) — yields nonlocal conservation laws and none yield any nonlocal symmetries.

Table 22. Nonlocal conservation laws for (mKdV-1)
$\varepsilon = \sqrt{\mp 1} = \mathrm{i}, 1$ respectively in the defocusing/focusing cases

Potential system	Ψ^t	Ψ^r	Remarks
$v_r = u,$ $v_t = u_{rr} - \varepsilon^2 u^3$	$e^{\varepsilon\sqrt{2}v}$	$\frac{1}{3}e^{\varepsilon\sqrt{2}v}(u^2 - \sqrt{2}u_r)$	$m = 0,\ p = 2$
$v_r = u,$ $v_t = u_{rr} + \frac{3}{2}r^{-1}u_r - \varepsilon^2 u^3$	$re^{\varepsilon\sqrt{2}v}$	$\frac{1}{3}e^{\varepsilon\sqrt{2}v}\left(ru^2 - \frac{\sqrt{2}}{2}(2ru_r + u)\right)$	$m = 3/2,\ p = 2$
$v_r = u,$ $v_t = u_{rr} + 3r^{-1}u_r - \varepsilon^2 u^3$	$r^2 e^{\varepsilon\sqrt{2}v}$	$\frac{1}{3}e^{\varepsilon\sqrt{2}v}(r^2u^2 - \sqrt{2}(r^2u_r + ru) + 1)$	$m = 3/2,\ p = 2$

Each of these nonlocal conservation laws gives rise to a further potential system. Of the three, none yield any additional nonlocal conservation laws, and only the one case $m = 0$ yields a nonlocal symmetry (previously found in [13,15]).

Nevertheless, these potential systems may be very useful for finding new exact solutions [4] of the equations (NLS), (dNLS), (dNLS-H), (mKdV-1,2,H), which we will pursue elsewhere.

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